# Aspects of self-organized criticality in a random driven interface model

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We introduce an external driven version of the solid-on-solid model of interface roughening in disordered media proposed by Leschhorn [H. Leschhorn, Physica A **195**, 324 (1993)]. The properties of the avalanches triggered by the external driving field are studied numerically in the interface dimension $\mathcal{D}=1$. It is found that just below the depinning transition the probability distributions of the characteristic quantities of the avalanches exhibit power-law behavior limited only by the system size. The exponents obtained for the probability distributions are discussed in the context of interface dynamics. [S1063-651X(98)06203-5]

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## I. INTRODUCTION

In recent years, a variety of studies were done to understand the interplay of interface depinning and self-organized criticality (SOC); see [1] and references therein. SOC behavior in driven systems is characterized by a statistical steady state far from equilibrium, which is produced by local initiations of avalanches where the probability distribution of their properties exhibit scale invariance; see [2] for a detailed discussion of SOC. Sandpile automata are well known prototype models exhibiting SOC [3]. In these models, particles are added to the system at randomly selected sites until a certain condition for stability is broken. Due to this instability an avalanche starts and it will be terminated if the condition for stability is fulfilled in the whole system again.

A typical model for the description of the interface motion in disordered systems is the Edwards-Wilkinson (EW) [4] equation with quenched disorder

$$\frac{\partial \mathsf{h}}{\partial t}=\vartheta\nabla^2\mathsf{h}+\eta(x,\mathsf{h})+F, \tag{1}$$

where the term $\vartheta\nabla^2\mathsf{h}$ reflects the smoothing surface tension while the uncorrelated quenched noise $\eta$ roughens the interface, $F$ represents an applied driving force. The interface motion in disordered systems is characterized by a so-called depinning transition that is based on the competition of the driving force and the disorder. Depending on the strength of the driving force the interface either gets pinned by the random impurities or it moves steadily with nonzero velocity. Based on dynamic scaling theory [5], which describes the scaling behavior of the correlation lengths parallel to the substrate,

$$\xi_{\parallel}\sim t^{1/z}, \tag{2}$$

and perpendicular to it,

$$\xi_{\perp}\sim\xi_{\parallel}^{\alpha}, \tag{3}$$

the interface motion can be described by local jumps from one pinning state to the next one. On a length scale $\xi\leqslant\xi_{\parallel}$ the interface will be moved forward by $\xi^{\alpha}$ over a time period $\xi^{z}$, where $\alpha$ denotes the roughness exponent and $z$ the dynamic exponent, respectively. In comparison with the sandpile models, these local movements between two pinning states can be named avalanches [6], whereas inside the avalanches the elements of the interface are correlated on the length scale $\xi$. Due to the statistical distribution of weak and strong pinning sites $\eta$ the avalanches have different sizes, whereas the maximum argument value of the probability distribution of the avalanche width is limited by $\xi_{\parallel}$. Since $\xi_{\parallel}$ diverges at the depinning transition, $\xi_{\parallel}\sim|F-F_C|^{-\nu}$ where $F_C$ is the critical driving force, one can expect that at the depinning transition the probability distributions of the characteristic avalanche quantities are scale invariant [7] so that a relation between self-organized criticality and the interface dynamics exists.

The depinning transition can be reached either by tuning the driving force to its critical value or by applying a constant velocity constraint. In the first case all sites of the interface are updated in parallel while in the second case the interface site with the largest local force, which is the right hand side of Eq. (1), is advanced only. This quasistatic approach to the depinning transition was studied recently [8,9] and it was found that this extremal dynamics leads to SOC. It is the purpose of the present paper to study the avalanchelike movements of an interface in disordered media in the vicinity of the depinning transition in a solid-on-solid (SOS) model [10], which reproduces the morphology of an interface described by Eq. (1) but not the dynamic behavior [11]. The definition of the model discussed here (see next section) suggests applying the formalism of randomly driven sandpile models to produce self-organized interfaces and not the quasi-static formalism.

## II. MODEL

The discussed model is defined as follows [10]. Each lattice point of the one-dimensional interface is assigned a random pinning force $\eta(x,\mathsf{h})$ that has the value $g$ with the probability $q$ and $-g$ with the probability $1-q$ where the parameter $g$ measures the strength of the pinning forces. Since for $q\neq 1/2$ the distribution of the noise $\eta$ is an unsymmetrical average driving force of $\langle F\rangle\simeq g(2q-1)$ is applied to the interface. The interface is defined by a set of integer

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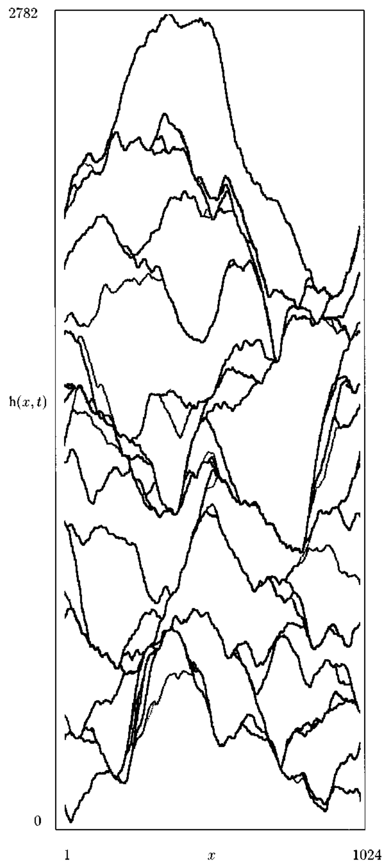


FIG. 1. A successive series of pinned interface, shown are the boundaries of the avalanches ($L=1024$).

column heights $\mathsf{h}(x,t)$ with the initial condition $\mathsf{h}(x,t=0)=0$ for $x=1,2,\ldots,L$. At each time step $t$ the sum of the discretized Laplacian and the noise

$$v(x)=\mathsf{h}(x-1,t)+\mathsf{h}(x+1,t)-2\mathsf{h}(x,t)+g\,\eta(x,\mathsf{h}) \tag{4}$$

is determined where periodic boundary conditions are assumed. The growth rule is defined by

$$\mathsf{h}(x,t+1)=\begin{cases}\mathsf{h}(x,t)+1 & \text{if } v(x)>0\\ \mathsf{h}(x,t) & \text{otherwise,}\end{cases} \tag{5}$$

where a parallel update is performed for all lattice points $x$. Additionally, every time a height $\mathsf{h}$ at position $x$ is increased a new value for $\eta(x,\mathsf{h})$ is drawn.

Obviously, if every local velocity $v(x)$ is smaller than or equal to zero the interface is pinned. If this condition is fulfilled, which arises only for $q<q_C$ where $q_C$ represents the critical concentration of the pinning forces, we choose randomly one position $x$. Now $\mathsf{h}(x)$ is increased by one unit and a new value for $\eta(x,\mathsf{h})$ is drawn. Due to this external driving an avalanche is started and the interface will be updated by the growing rule Eq. (5) until the interface gets pinned again so that the avalanche has stopped; see Fig. 1 for a series of the avalanche boundaries. Note that such an external driving of a pinned interface in a model of imbibition in porous media was performed by Barabási *et al.* [12].

After the relaxation process has stopped the characteristic quantities of the avalanche are measured. The first one is the area of the avalanche

$$a=\sum_{x=1}^{L}\mathsf{h}(x,t_f)-\mathsf{h}(x,t_i), \tag{6}$$

where $\mathsf{h}(x,t_i)$ and $\mathsf{h}(x,t_f)$ denote the initial and the final interface profile of each avalanche, respectively. The next quantity is the duration of the avalanche $\tau=t_f-t_i$, which is the time the interface needs to relax from one pinning state to the next one. Additionally we measured the height $h$ of the avalanche defined by the maximum value of $\mathsf{h}(x,t_f)-\mathsf{h}(x,t_i)$ for all $x$ and the width $w$ of the avalanche, which is that length scale at which the values of $\mathsf{h}(x,t_f)$ and $\mathsf{h}(x,t_i)$ differ from each other. Besides the maximum norm used here for the characterization of the height and the width of an avalanche, the radius of gyration can be used. However, in preliminary examinations of these avalanche quantities no different scaling behavior was obtained. Thus, due to the lower numerical expense only the maximum norm of $h$ and $w$ will be analyzed in the following.

## III. RESULTS

Approaching the depinning transition from below, the typical fluctuations of the interface diverge so that only at the depinning transition the whole interface is scale invariant. Since the avalanches describe the interface movements between two consecutive pinning states one can expect that the probability distributions of the characteristic avalanche quantities are likewise scale invariant only at or very close to the depinning transition, which is verified by the analysis of the field dependence of the correlation length (see below).

To examine a possible SOC behavior of the system we measured the probability density $P(X)$ for different system sizes $128\leqslant L\leqslant 8192$ where the symbol $X$ represents one of the four quantities, area $a$, duration $\tau$, height $h$, or width $w$, which characterize the avalanche. The system parameters are $g=1$ and $q=0.799$, which is very close to the critical concentration $q_C\simeq 0.8007$ [11] where the depinning transition occurs. The above defined quantities were averaged over $N\approx 10^9/L$ avalanches. The probability densities of the characteristic quantities exhibit power-law behavior

$$P_X(X)\propto X^{-\gamma_X} \tag{7}$$

up to a sharp cutoff length which depends on the system size $L$; see Figs. 2(a)–2(d). The exponents $\gamma_X$ are obtained by power-law fits over the straight portions of the data; see Table I for the corresponding values. For large system sizes $L$ all data show a weak positive bend on logarithmic scales. Since an independent determination of the exponents $\gamma_X$, which can be represented by the correlation length exponent $\nu$, the roughness exponent $\alpha$, and the dynamic exponent $z$ leads to similar results (see below) this deviation from a perfect power-law behavior may be traced back onto an insufficient averaging of the probability distributions due to computer limitations.

The scale invariance of the probability distributions is characterized by an algebraic increase of the cutoff value of

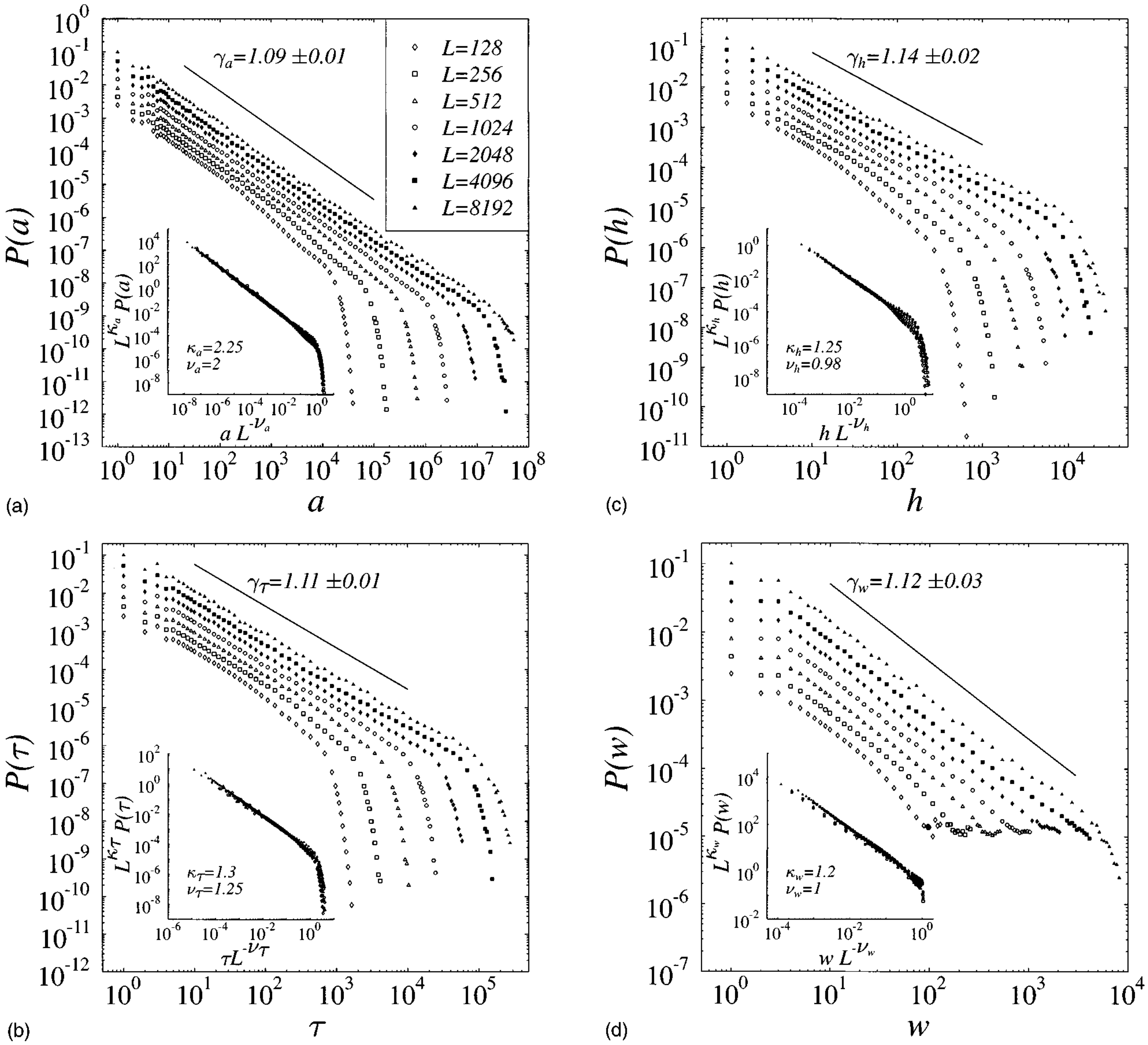


FIG. 2. The probability distribution of (a) the area of the avalanches $P(a)$, (b) the duration of the avalanches $P(\tau)$, (c) the height of the avalanches $P(h)$, and (d) the width of the avalanches $P(w)$ for the system sizes $L=128,\ldots,8192$, symbols see (a). The curves for $L<8192$ are shifted in the downward direction. The solid lines represent corresponding fits according to Eq. (7) with the indicated exponents. The insets in the figures show the corresponding finite size plots according to Eq. (8).

TABLE I. The values of the exponents describing the probability densities $P(X)$ of the quantities that characterized the avalanches where $X$ denotes the area $a$, the duration $\tau$, the height $h$, and the width $w$. The errors of the exponents $\gamma_X$ are given by least-square fits according to Eq. (7) while the errors of the exponents $\kappa_X$ and $\nu_X$ are determined by scaling plots as discussed in the main text.

| $X$ | $\gamma_X$ | $\kappa_X$ | $\nu_X$ |
|---|---|---|---|
| $a$ | $1.09\pm0.01$ | $2.4\pm0.12$ | $2.0\pm0.1$ |
| $\tau$ | $1.11\pm0.01$ | $1.6\pm0.08$ | $1.25\pm0.06$ |
| $h$ | $1.14\pm0.02$ | $1.3\pm0.07$ | $0.98\pm0.05$ |
| $w$ | $1.12\pm0.03$ | $1.2\pm0.06$ | $1.0\pm0.05$ |

the probability distributions. To study this characteristic we performed a finite-size scaling analysis [13] according to

$$P_X(X,L)=L^{-\kappa_X}f_X(L^{-\nu_X}X). \tag{8}$$

The scaling plots are shown in the insets of Figs. 2(a)–2(d) and the corresponding values of the exponents are listed in Table I. In order that this ansatz has to describe the algebraic decay of Eq. (7) the universal function $f(y)$ has to scale like $f(y)\propto y^{-\gamma_X}$ for $y\ll 1$. Furthermore the three exponents $\kappa_X$, $\nu_X$, and $\gamma_X$ have to fulfill the relation

$$\kappa_X=\nu_X\gamma_X. \tag{9}$$

The values of the exponents $\kappa_X$ and $\nu_X$ are determined by scaling plots. Changing these values approximately by $\pm5\%$ the quality of the corresponding scaling plot is not changed

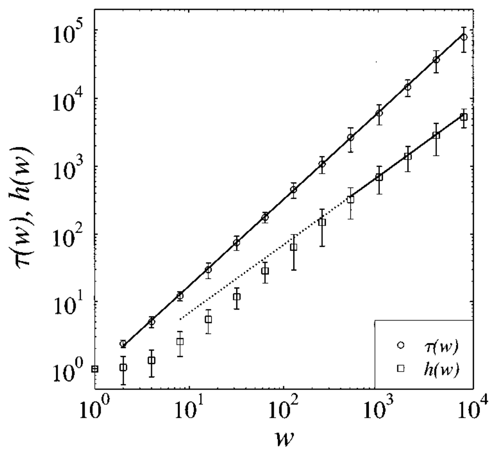


FIG. 3. The avalanche duration $\tau(w)$ and the avalanche height $h(w)$ as a function of the avalanche width $w$. The error bars of $\tau$ and $w$ are given by the width of the distribution functions of $\tau$ and $h$ at constant $w$, respectively. The lines represent fits according to $\tau(w)\sim w^z$ and $h(w)\sim w^\alpha$ with $z=1.28\pm0.03$ and $\alpha=1.01\pm0.02$. The dashed line represents an extrapolation of the fitting curve $h(w)$ to elucidate the deviation from the asymptotic scaling behavior for $w<512$.

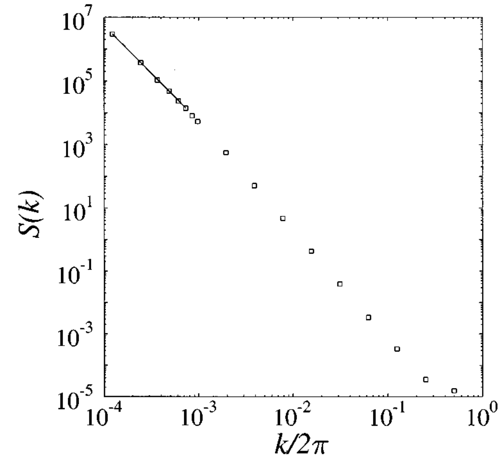


FIG. 4. The structure factor $S(k)$ of the pinned interface positions $\mathsf{h}(x,t_f)$ as a function of $k$ for $L=8192$. The solid line represents a fit according to Eq. (11) with $\tilde{\gamma}=2.99$.

dramatically, so that the error of these exponents is approximately $\pm 5\%$. Thus, with these values for the errors of the exponents the scaling relations Eq. (9) are fulfilled.

In the following we will show that the exponents $\nu_X$ that describe the system size dependence of the cutoff lengths in the probability distributions of the characteristic avalanche quantities can be described in terms of the dynamic exponent $z$ and the roughness exponent $\alpha$, which characterize the scaling behavior of the two correlation lengths $\xi_\parallel$ and $\xi_\perp$. These two quantities measure the width and the height of the typical fluctuations of the interface so that it is suggested to identify these quantities with the width $w$ and the height $h$ of the avalanches, respectively. With $\xi_\parallel\sim L$, which is based on the dynamic scaling hypothesis [5] it follows that $\nu_w=1$ and with Eq. (3) the identity $\nu_h=\alpha$ follows directly. Recently, for the automaton model of Leschhorn the roughness exponent $\alpha_d\simeq1$ was obtained just above the depinning transition [11] in agreement with the value obtained here. The dependence of $\xi_\perp$ on $\xi_\parallel$ according to Eq. (3) and the dependence of the avalanche height $h$ on the avalanche width $w$ can be verified by the measurement of $h(w)$; see Fig. 3. For $w<512$ a value $\alpha\simeq1.17$ is obtained for the roughness exponent while for larger values of $w$ one obtains a value $\alpha=1.01\pm0.02$, which is in good agreement with the result of the scaling plot. To check the asymptotic scaling behavior of $h(w)$ we measure the structure factor [14]

$$S(k)=\langle\hat{\mathsf{h}}(k,t_f)\hat{\mathsf{h}}(-k,t_f)\rangle \tag{10}$$

with $\hat{\mathsf{h}}(k,t_f)=L^{-1/2}\Sigma_x[\mathsf{h}(x,t_f)-\langle\mathsf{h}(x,t_f)\rangle]e^{ikx}$. For small $k$ the structure factor scales as

$$S(k)\propto k^{-\tilde{\gamma}}=k^{-(2\alpha+1)}, \tag{11}$$

see Fig. 4. The data exhibit an unexpected turning point at $k\approx2\pi/1024$ so that only the smallest $k$ modes have to be taken into account to obtain the asymptotic $(k\to0)$ behavior. From a power-law fit of the six smallest $k$ modes we obtained for the roughness exponent $\alpha\simeq1$, in agreement with the above analysis. Furthermore, fitting the structure factor in an intermediate $k$ regime with $k>2\pi/1024$ one obtains for the roughness exponent $\alpha=1.22\pm0.01$, which explains the nonasymptotic scaling behavior of $h$ for small values of $w$. Due to the definition of the model discussed here the avalanches are compact objects, i.e., it can be assumed that the area of an avalanche scales like [9]

$$a\sim\xi_\parallel\xi_\perp \tag{12}$$

and therefore $\nu_a=\nu_w+\nu_h=1+\alpha$. Based on the dynamic scaling hypothesis, in the time $\tau$ a fluctuation of the wavelength $\xi_\parallel^z$ occurs. Therefore, the exponent $\nu_\tau$ that describes the system size dependence of the maximum avalanche duration can be identified with the dynamic exponent $z$. This identification is confirmed by a measurement of the avalanche duration as a function of the avalanche width $\tau(w)\sim w^z$ with $z=1.28\pm0.03$; see Fig. 3. The value of the dynamic exponent obtained here $(z\simeq1.25)$ is smaller than the value that was obtained just above the depinning transition $(z_d\simeq1.5)$ [11]. Thus, due to the additional external disturbance introduced here the dynamic behavior is changed. Note that different dynamic behaviors for interface models and their external driven versions are observed earlier [6].

As shown exemplary in Fig. 3 for the dependence of the avalanche duration and the avalanche height on the avalanche width, the characteristic avalanche quantities depend on each other in a statistical manner. Thus, due to the fundamental transformation law of probabilities

$$P_X(X)dX=P_Y(Y)dY, \tag{13}$$

the exponents $\nu_X$ and therefore the roughness exponent $\alpha$ and the dynamic exponent $z$ can be expressed in terms of the exponents $\gamma_X$ and $\gamma_Y$. For instance, from $h\sim w^\alpha$ it follows that

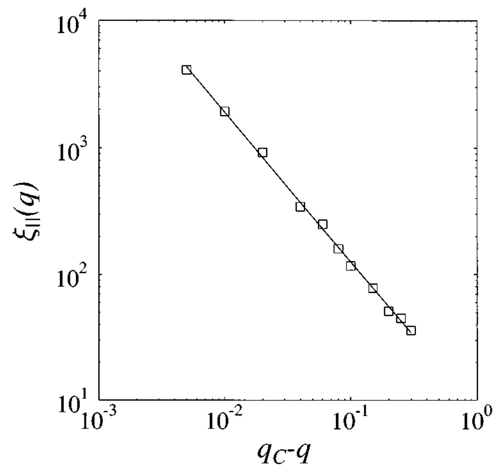


FIG. 5. The $q$ dependence of the parallel correlation length $\xi_{\|}$ where the solid line represents a fit according to $\xi_{\|}\sim(q_C-q)^{-\nu}$ leading to $\nu=1.17\pm0.02$.

$$h^{-\gamma_h}dh=w^{-\gamma_h\alpha}\frac{dh}{dw}dw=w^{-\gamma_w}dw, \tag{14}$$

and thus $\alpha=(\gamma_w-1)/(\gamma_h-1)$. Along with Eq. (9) it is therefore possible to calculate the whole exponents describing the finite-size behavior of the characteristic quantities of the avalanches if one of the eight exponents $\gamma_X$ or $\kappa_X$, respectively, and the roughness exponent as well as the dynamic exponent are known. For the conjecture that the probability distribution of the width of an avalanche obeys the scaling form

$$P(w)\simeq\frac{1}{w^{\gamma}}\rho(w/\xi_{\|}) \tag{15}$$

with $\xi_{\|}\propto(F_C-F)^{-\nu}$ near the depinning transition Narayan and Fisher [15] obtained with scaling arguments for the EW equation with quenched disorder $\gamma=\mathcal{D}+1-1/\nu$. Furthermore they obtained $\nu=1/(2-\alpha)$ so that no new independent exponent arises in the description of the avalanche properties in the EW model. At the depinning transition ($F_C=F$) the only characteristic length scale is the system length $L$ and $\xi_{\|}$ in Eq. (15) can be replaced by $L$. Thus the exponent $\gamma_w$ can be identified with $\gamma$, which leads to $\gamma_w\simeq1$ and along with $\alpha\simeq1$ to $\gamma_X\simeq1$ for the remaining exponents $\gamma_X$. However, these results are in opposition to the numerical values that yield to values larger than one, even if one takes the error bars of the exponents into consideration.

The determination of the driving field dependence of the correlation length $\xi_{\|}\sim(q_C-q)^{-\nu}$, which is given by the driving field dependence of the maximum value of the avalanche width results in $\nu=1.17\pm0.02$ for the correlation length exponent $\nu$, see Fig. 5. This analysis shows that the scale invariance of the probability distributions of the characteristic avalanche quantities exists only at the depinning transition. Furthermore, it shows that the discrepancy of the numerical results $\gamma_X\simeq1.1$ with the theoretical expectations can be traced back onto the fact that the exponent relation $\nu=1/(2-\alpha)$ obtained for the EW model is not right here. As discussed above the correlation length exponent $\nu$ corresponds to the exponent $\gamma_w$, $\gamma_w=2-1/\nu=1.15\pm0.02$. Furthermore, using Eq. (13) one obtains $\gamma_h=1+(\gamma_w-1)/\alpha=1.15\pm0.03$, $\gamma_\tau=1+(\gamma_w-1)/z=1.12\pm0.04$, and $\gamma_a=1+(\gamma_w-1)/(\alpha+1)=1.08\pm0.01$. Therefore, the calculated values $\gamma_X$ are in agreement with the values obtained directly with Eq. (7). Since the exponents $\nu_X$ are determined also by $\alpha$ and $z$ the finite-size behavior of the probability distributions of the characteristic avalanche quantities is determined completely by the exponents $\alpha$, $z$, and $\nu$ so that no new exponents have to be introduced to describe the behavior of the avalanches.

## IV. SUMMARY

To summarize, we have shown that a local disturbance of the pinned interface described by the automaton model of Leschhorn triggers an avalanche whereas the probability distributions of the characteristic avalanche quantities are scale invariant. A detailed analysis shows that the probability distributions are determined completely by the roughness exponent $\alpha$, the dynamic exponent $z$, and the correlation length exponent $\nu$. Furthermore, the analysis of the driving field dependence of the correlation length shows that the correlation length exponent $\nu$ is not given by the scaling relation $\nu=1/(2-\alpha)$, which is exact for the Edwards-Wilkinson model with quenched disorder. Thus the automaton model of Leschhorn does not belong to the Edwards-Wilkinson universality class, which confirms the results presented earlier [11].

Following [16] the characteristic property of self-organized critical systems is the occurrence of scale-invariant avalanches triggered by disturbances. In the context discussed here the term self-organization means that after an external disturbance a system turns back into a stable state. Thus due to the obtained scale-invariant probability distributions of the characteristic avalanche properties the interface model discussed here shows self-organized criticality. However, a stricter formulation of SOC [2] requires that the scale invariance of the quantities are generic and are not achieved by fine tuning of a relevant parameter. In the model discussed here scale invariance is observed only at the depinning transition, which is achieved by choosing the critical concentration $q_C$ of the pinning sites. Thus the occurrence of SOC is only given in the weaker sense of [16].

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